Local rigidity of homogeoenous actions of parabolic subgroups of rank-one Lie groups

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Abstract

We show the local rigidity of the natural action of the Borel subgroup of $SO_+(n,1)$ on a cocompact quotient of $SO_+(n,1)$ for $n \geq 3$.

1 Introduction

Rigidity theory of actions of non-compact groups has been rapidly developed in the last two decades. It is found that many actions related to Lie groups of real-rank greater than one exhibit rigidity. See Fisher's survey paper [2], for example. However, there are only few results on actions related to Lie groups of real-rank one. The aim of this paper is to show the local rigidity of some natural actions related to such groups.

Let G be a Lie group and M be a C^∞ manifold. By $C^\infty(M \times G, M)$, we denote the space of C^∞ maps from $M \times G$ to M with the compact-open C^∞-topology. Let $\mathcal{A}(M,G)$ be the set of C^∞ right actions of G on M. It is a closed subset of $C^\infty(M \times G, M)$. We say two actions $\rho_1 : M_1 \times G \rightarrow M_1$ and $\rho_2 : M_2 \times G \rightarrow M_2$ are *C^∞-conjugate* if there exists a C^∞ diffeomorphism h and an automorphism σ of G such that $h(\rho_1(x,g)) = \rho_2(h(x), \sigma(g))$ for any $x \in M_1$ and $g \in G$. An action $\rho \in \mathcal{A}(M,G)$ is called *C^∞-locally rigid* if the C^∞-conjugacy class of ρ is a neighborhood of ρ in $\mathcal{A}(M,G)$. We say an action $\rho \in \mathcal{A}(M,G)$ is *locally free* if the isotropy subgroup $\{g \in G \mid \rho(x,g) = x\}$ is a discrete subgroup of g for any $x \in M$.

Let H be its closed subgroup of a Lie group G and Γ a cocompact lattice of G. We define *the standard H-action* ρ_0 on $\Gamma\backslash G$ by $\rho_0(\Gamma g, h) = \Gamma(gh)$. It is a locally free action. We say an action is *homogeneous* if it is C^∞-conjugate to the standard action associated with some cocompact lattice.

Suppose that the Lie group G is connected and semi-simple. Let $G = KAN$ be its Iwasawa decomposition. The dimension of the abelian subgroup A is called *the real-rank* of G. Let M be the centralizer of A in K. The group $P =$

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MAN is called *the Borel subgroup* associated with the Iwasawa decomposition $G = KAN$. It is known that the conjugacy class of the Borel subgroup does not depend on the choice of the Iwasawa decomposition. When $G = SL(2, \mathbb{R})$ for example, a Borel subgroup P is conjugate to the group GA of the upper triangular matrices in $SL(2, \mathbb{R})$.[1] Fix a cocompact lattice Γ of $SL(2, \mathbb{R})$ and put $M_\Gamma = \Gamma \backslash SL(2, \mathbb{R})$. Let ρ_0 be the standard P-action on M_Γ. It is not locally rigid since deformation of lattice Γ gives a non-trivial deformation of actions. However, there are several rigidity results on ρ_0 Ghys [3] proved that if a locally free GA-action on M_Γ admits an invariant volume, then it is homogeneous. In [4], he remove the assumption on invariant volume when $H^1(M_\Gamma)$ is trivial. The author of this paper [1] classified all locally free action of P on M_Γ up to C^∞-conjugacy without any assumption. As a consequence, there exists a non-homogeneous locally free GA-action on M_Γ when $H^1(M_\Gamma)$ is non-trivial. In a forthcoming paper, he will also show that the standard GA-action ρ_0 admits a C^∞ deformation into non-homogeneous actions in this case.

By Mostow's rigidity theorem, any deformation of a cocompact lattice is trivial if G is a higher-dimensional Lie group of real-rank one. So, it is natural to ask whether the standard P-action is C^∞-locally rigid or not in this case. The main result of this paper answers this question when G is $SO_+(n, 1)$.

Theorem 1.1. *Let P be a Borel subgroup of $SO_+(n, 1)$ and Γ be a torsion-free cocompact lattice of $SO_+(n, 1)$. If $n \geq 3$, then the standard P-action on $\Gamma \backslash SO_+(n, 1)$ is C^∞-locally rigid.*

To ending the introduction, we remark on the local rigidity of the orbit foliation. Let $\mathcal{F}_\Gamma$ be the orbit foliation of the standard P-action on $\Gamma \backslash SO_+(n, 1)$. Ghys [5] showed a global rigidity result of $\mathcal{F}_\Gamma$ for $n = 2$.For $n \geq 3$, Yue [11] proved a partial result and Kanai [6] claimed the local rigidity of $\mathcal{F}_\Gamma$. However, Kanai's proof contains a serious gap[2] and it is not fixed so far. Hence, the local rigidity of $\mathcal{F}_\Gamma$ is still open. If any foliation sufficiently close to $\mathcal{F}_\Gamma$ carries an action of P, then the local rigidity of $\mathcal{F}_\Gamma$ follows from our theorem.

2 Preliminaries

In this section, we introduce some notations and review several known facts which we will use in the proof of Theorem 1.1.

2.1 The group $SO_+(n, 1)$

Fix $n \geq 3$ and let $I_{n,1}$ be the diagonal matrix of size $(n + 1)$ whose diagonal elements are $1, \ldots, 1, -1$. Let $SO_+(n, 1)$ be the identity component of the subgroup of $GL(n + 1, \mathbb{R})$ consisting of matrices A satisfying ${}^{\tau}\!A I_{n,1} A = I_{n,1}$. For

[1] It is isomorphic to the Group of orientation preserving Affine transformations of the real line.

[2] The C^1-regularity of the strong unstable foliation claimed in the last sentence of p.677 does not hold in general.

any $1 \leq n' \leq n$, the standard embedding $GL(n', \mathbb{R}) \hookrightarrow GL(n+1, \mathbb{R})$ induces an embedding of $SO(n')$ into $SO_+(n, 1)$.

Let $\mathfrak{so}(n, 1)$ be the Lie algebra of $SO_+(n, 1)$. By E_{ij}, we denote the square matrix of size $(n+1)$ such that the (i, j)-entry is one and other entries are zero. Put $X = E_{n(n+1)} - E_{(n+1)n}$, $Y_i = (E_{i(n+1)} + E_{(n+1)i}) - (E_{in} - E_{ni})$, and $Y_i' = (E_{i(n+1)} + E_{(n+1)i}) + (E_{in} - E_{ni})$ for $i, j = 1, \cdots, n$. Then, $\mathfrak{so}(n, 1)$ is generated by $X, Y_1, \cdots, Y_{n-1}, Y_1', \cdots, Y_{n-1}'$ and the Lie subalgebra corresponding to the subgroup $SO(n-1)$ of $SO_+(n, 1)$. It is easy to check that

$$[Y_i, X] = -Y, \quad [Y_i', X] = Y_i', \quad [Y_i, Y_j] = [Y_i', Y_j'] = 0 \tag{1}$$

for any $i, j = 1, \cdots, n-1$, and

$$\begin{aligned} \mathrm{Ad}_m(X) &= X \\ \mathrm{Ad}_m(Y_1, \cdots, Y_{n-1}) &= (Y_1, \cdots, Y_{n-1}) \cdot m \end{aligned} \tag{2}$$

and for any $m \in SO(n-1)$.

Let $SO_+(n, 1) = KAN$ be the Iwasawa decomposition of $SO_+(n, 1)$ associated with the involution $\theta_0 : g \mapsto ({}^\tau g)^{-1}$. Then, $K = SO(n)$ and, A and N are the subgroups of $SO_+(n, 1)$ corresponding to the Lie subalgebras spanned by X and $\{Y_1, \ldots Y_{n-1}\}$, respectively. Since the centralizer of A in $SO(n)$ is $SO(n-1)$, the Borel subgroup corresponding to θ_0 is $SO(n-1)AN$.

2.2 Anosov flows

A C^1 flow Φ on a closed manifold M is called *Anosov*, if it has no stationary points and there exists a continuous splitting $TM = T\Phi \oplus E^{ss} \oplus E^{uu}$, a constant $\lambda > 0$, and a continuous norm $\|\cdot\|$ on TM which satisfy the following properties:

- $T\Phi$ is the one-dimensional subbundle tangent to the orbit of Φ.
- E^{ss} and E^{uu} are $D\Phi$-invariant subbundles.
- $\|D\Phi^t(v^s)\| \leq e^{-\lambda t}\|v^s\|$ and $\|D\Phi^t(v^u)\| \geq e^{\lambda t}\|v^u\|$ for any $v^s \in E^{ss}$, $v^u \in E^{uu}$, and $t \geq 0$.

The subbundles E^{ss}, E^{uu}, $T\Phi \oplus E^{ss}$, and $T\Phi \oplus E^{uu}$ are called *the strong stable, strong unstable, weak stable*, and *weak unstable subbundles*, respectively. It is known that they generate continuous foliations with C^r leaves, if Φ is a C^r flow. The foliations are called *the strong stable foliation, etc.*

The following proposition may be well-known for experts, but we give a proof for convenience of the readers.

Proposition 2.1. *Let Φ_1 and Φ_2 be Anosov flows on a closed manifold M. Suppose that Φ_1 and Φ_2 have the common strong unstable foliation $\mathcal{F}^{uu}$ and $\mathcal{F}^{uu}(\Phi_1^t(x)) = \mathcal{F}^{uu}(\Phi_2^t(x))$ for any $x \in M$ and $t \in \mathbb{R}$. Then, there exists a homeomorphism h of M such that $\Phi_2^t \circ h = h \circ \Phi_1^t$ for any $t \in \mathbb{R}$ and $h(\mathcal{F}^{uu}(x)) = \mathcal{F}^{uu}(x)$ for any $x \in M$.*

Proof. Let $\mathcal{H}$ be the set of continuous maps $h : M \to M$ which preserves each leaf of $\mathcal{F}^{uu}$. Fix a Riemannian metric g of M. Let d^u be the leafwise distance on leaves of $\mathcal{F}^{uu}$ which is determined by the restriction of the metric g to each leaf. We define a distance d on $\mathcal{H}$ by $d(h, h') = \sup_{x \in N_\Gamma} d^u_\Gamma(h(x), h'(x))$. It is a complete metric on $\mathcal{H}$.

For $i, j \in \{1, 2\}$, we define continuous flows Θ_{ij} on $\mathcal{H}$ by $\Theta^t_{ij}(h) = \Phi_i^{-t} \circ h \circ \Phi^t_j$. Since Φ_i and Φ_j expand $\mathcal{F}^{uu}$ uniformly, Θ^t_{ij} is a uniform contraction for any sufficiently large $t > 0$. By the contracting mapping theorem, there exists a unique fixed point $h_{ij} \in \mathcal{H}$ of the flow Θ_{ij}. Since both $h_{ij} \circ h_{ji}$ and the identity map of M are fixed point of Θ_{ii}, $h_{ij} \circ h_{ji}$ is the identity map for $i, j \in 1, 2$. In particular, h_{ij} is the inverse of h_{ji}. Therefore, h_{21} is a homeomorphism in $\mathcal{H}$ such that $h_{21} \circ \Phi^t_1 = \Phi^T_2 \circ h_{21}$ for any $t \in \mathbb{R}$. $\square$

Let Ψ be a flow on a manifold M. A C^∞ function α on $M \times \mathbb{R}$ is *a cocycle over* Ψ if $\alpha(x, 0) = 0$ and $\alpha(x, t + t') = \alpha(x, t) + \alpha(\Psi^t(x), t')$ for any $x \in M$ and $t, t' \in \mathbb{R}$. We say Ψ is *topologically transitive* if there exists $x_0 \in M$ whose orbit $\{\Psi^t(x_0) \mid t \in \mathbb{R}\}$ is a dense subset of M.

Theorem 2.2 (The C^∞ Livschitz Theorem [7])**.** *Let Φ be a C^∞ topologically transitive Anosov flow on a closed manifold M and α be a C^∞ cocycle over Φ. If $\alpha(x, T) = 0$ for any $(x, t) \in M \times \mathbb{R}$ satisfying $\Phi^T(x) = x$, then there exists a C^∞ function β on M such that $\alpha(x, t) = \beta(\Phi^t(x)) - \beta(x)$ for any $x \in M$ and $t \in \mathbb{R}$. Moreover, if α is sufficiently C^∞-close to 0, then we can choose β so that it is C^∞-close to 0.*

We say an Anosov flow Φ is *s- (resp. u-)conformal* if $D\Phi^t$ is conformal on $E^{ss}(x)$ (*resp.* $E^{uu}(x)$) for any $x \in M$ with respect to some continuous metric[3] on E^{ss}. The following result plays fundamental role in the proof of Theorem 1.1.

Theorem 2.3 (de la Llave [8])**.** *Let Φ_1 and Φ_2 be C^∞ s-conformal topologically transitive Anosov flows on a closed manifold M. For $i = 1, 2$, let $\mathcal{F}^{ss}_i$ be of Φ_1. Suppose that the dimensions of the strong stable foliation of Φ_1 and Φ_2 are greater than one. If a homeomorphism h of M satisfies $\Phi^t_2 \circ h = h \circ \Phi^t_1$ for any $t \in \mathbb{R}$, then the restriction of h to a leaf of the strong stable foliation of Φ_1 is a C^∞ diffeomorphism to a leaf of the strong stable foliation of Φ_2. Moreover, if both Φ_1 and Φ_2 are u-conformal in addition, then h is a C^∞ diffeomorphism of M.*

We say that an Anosov flow is *contact* if it preserves a C^1-contact structure. It is easy to see that any contact structure invariant under an Anosov flow is the direct sum of the strong stable subbundle and the strong unstable subbundle.

Proposition 2.4. *Let Φ be a contact Anosov flow on a closed manifold M. If Φ is s-conformal, then it is u-conformal.*

[3]We may assume that the metric is Hölder continuous and C^∞ along leaves of the strong stable foliation. See Sadovskaya [10].

Proof. Let $TM = T\Phi \oplus E^{ss} \oplus E^{uu}$ be the Anosov splitting of Φ and X be the vector field generating Φ. Take a continuous metric g_- such that Φ is s-conformal with respect to g_-. Let α be one-form α on M such that $\mathrm{Ker}\,\alpha = E^{ss} \oplus E^{uu}$ and $\alpha(X) = 1$. Since $E^{ss} \oplus E^{uu}$ is a Φ-invariant C^1-contact structure, the one-form α is a C^1-contact form invariant under Φ. Hence, $\omega = d\alpha$ is a $D\Phi$-invariant two-form which is non-degenerate on $E^{ss} \oplus E^{uu}$.

By the invariance, $\omega(v, v') = \omega(D\Phi^t(v), D\Phi^t(v'))$ for $x \in M$, $v, v' \in E^{ss}(x)$, and $t \in \mathbb{R}$, and the latter converges to zero as $t \to +\infty$. Hence, the restriction of ω to E^{ss} is zero. The restriction of ω to E^{uu} also is.

We define a metric g_+ on E^{uu} by

$$g_+(v, v') = \sum_{j=1}^{n} \omega(u_i, v)\omega(u_i, v')$$

for $v, v' \in E^{uu}(x)$, where $(u_1, \ldots, u_n)$ is an orthonormal basis of $E^{ss}(x)$ with respect to g_-. By a direct calculation, we can check that g_+ does not depend on the choice of the orthonormal basis $(u_1, \ldots, u_n)$. Remark that g_+ is a continuous metric on E^{uu}.

Fix $x \in M$ and $t \in \mathbb{R}$. Since Φ is s-conformal, there exists a real number $a \neq 0$ and orthonormal basis $(u_1, \ldots, u_n)$ of $E^{ss}(x)$ and $(u'_1, \ldots, u'_n)$ of $E^{ss}(\Phi^t(x))$ such that $D\Phi^t(u_i) = a \cdot u'_i$ for any $i = 1, \cdots, n-1$. Then,

$$\omega(u'_i, D\Phi^t(v)) = a^{-1}\omega(D\Phi^t(u_i), D\Phi^t(v)) = a^{-1}\omega(u_i, v)$$

for any $v \in E^{uu}(x)$. It implies that $g_+(D\Phi^t(v), D\Phi^t(v')) = a^{-2}g_+(v, v')$ for any $v, v' \in E^{uu}(x)$. Therefore, $D\Phi$ is u-conformal with respect to g_+. $\square$

3 Proof of Theorem 1.1

Let $SO_+(n, 1) = SO(n)AN$ be the Iwasawa decomposition and $P = SO(n-1)AN$ the Borel subgroup of $SO_+(n, 1)$ described in Section 2.1. Fix a torsion-free cocompact lattice Γ of $SO_+(n, 1)$, and put $M_\Gamma = \Gamma\backslash SO_+(n, 1)$ and $N_\Gamma = \Gamma\backslash SO_+(n, 1)/SO(n-1)$. We denote the natural projection from M_Γ to N_Γ by π. Let ρ_0 be the standard P-action on M_Γ. For $x = \Gamma g \in M_\Gamma$ and $m \in SO(n-1)$, we put $x \cdot m = \rho_0(x, m) = \Gamma(gm)$.

For any $\rho \in \mathcal{A}(M_\Gamma, P)$ and $g \in P$, we define a C^∞ diffeomorphism ρ^g of M_Γ by $\rho^g(x) = \rho(x, g)$.

3.1 Induced Anosov flows on N_Γ

Let $\mathcal{A}_*(M_\Gamma, P)$ be the set of locally free P-actions on M_Γ which satisfy $\rho(x, m) = x \cdot m$ for any $x \in M_\Gamma$ and $m \in SO(n-1)$.

Proposition 3.1. *If $\rho : M_\Gamma \times P \to M_\Gamma$ is sufficiently C^∞-close to ρ_Γ then ρ is C^∞ conjugate to an action in $\mathcal{A}_*(M_\Gamma, P)$ which is C^∞-close to ρ_Γ.*

Proof. It is an immediate corollary of Palais' stability theorem of compact group action ([9]). □

For $\rho \in \mathcal{A}_*(M_\Gamma, P)$, we define a flow Φ_ρ on N_Γ by $\Phi_\rho^t(\pi(x)) = \pi(\rho^{\exp(tX)}(x))$. It is well-defined since $\exp(tX)$ commutes with any element of $SO(n-1)$. We call the flow Φ_ρ *the flow* induced by ρ.

For $\rho \in \mathcal{A}_*(M_\Gamma)$, we define vector fields $Y_1^\rho, \dots, Y_{n-1}^\rho$ on M_Γ by $Y_i^\rho(x) = (d/dt)\rho^{\exp tY_i}(x)|_{t=0}$.

Lemma 3.2. *For any $x \in M_\Gamma$ and $m \in SO(n-1)$,*

$$(D\pi(Y_1^\rho(x \cdot m)), \cdots, D\pi(Y_{n-1}^\rho(x \cdot m)) = (D\pi(Y_1^\rho(x)), \cdots, D\pi(Y_{n-1}^\rho(x))) \cdot m.$$

Proof. For any $x \in M_\Gamma$, $t \in \mathbb{R}$, and $m \in SO(n-1)$,

$$\begin{aligned}\pi \circ \rho(x \cdot m, \exp(tY_i)) &= \pi(\rho(x, [m \exp(tY_i) m^{-1}]) \cdot m) \\ &= \pi \circ \rho(x, \exp(t \cdot \mathrm{Ad}_m(Y_i))).\end{aligned}$$

Hence, the equation (2) implies the lemma. □

By the above lemma, we can define a C^∞ subbundle E_ρ^- of TN_Γ by

$$E_\rho^-(\pi(x)) = D\pi(\langle Y_1^\rho(x), \dots Y_{n-1}^\rho(x)\rangle).$$

There exists a C^∞ metric g_ρ on E_ρ^- such that $(D\pi(Y_1^\rho(x)), \dots, D\pi(Y_{n-1}^\rho(x)))$ is an orthonormal basis of $E_\rho^-(x)$ with respect to g_ρ. The subbundle E_ρ^- is $D\Phi_\rho$-invariant and

$$\|D\Phi_\rho^t(v)\|_{g_\rho} = e^{-t}\|v\|_{g_\rho} \tag{3}$$

for any $t \in \mathbb{R}$ and $v \in E_\rho^-$.

For $i = 1, \cdots, n-1$, let Y_i' be a vector field on M_Γ given by $Y_i^+(x) = (d/dt)x\exp(tY_i')|_{t=0}$. Similar to the above, we can define a C^∞ subbundle $E_{\rho_0}^+$ of TN_Γ and its C^∞ metric g^+ such that

$$E_{\rho_0}^+(\pi(x)) = D\pi(\langle Y_1^+(x), \dots Y_{n-1}^+(x)\rangle)$$

and $(D\pi(Y_1^+(x)), \dots, D\pi(Y_{n-1}^+(x)))$ is an orthonormal basis of $E_{\rho_0}^+(x)$ with respect to g^+. The subbundle $E_{\rho_0}^+$ is $D\Phi_{\rho_0}$-invariant and

$$\|D\Phi_{\rho_0}^t(v'))\|_{g^+} = e^t\|v'\|_{g^+} \tag{4}$$

for any $t \in \mathbb{R}$ and $v' \in E_{\rho_0}^+$. The flow Φ_{ρ_0} is an Anosov flow with the Anosov splitting $TN_\Gamma = T\Phi \oplus E_{\rho_0}^- \oplus E_{\rho_0}^+$ and it is s- and u-conformal with respect to g_{ρ_0} and g^+, respectively. It is known that $E_{\rho_0}^- \oplus E_{\rho_0}^+$ is a Φ_{ρ_0}-invariant contact structure.

Since the set of Anosov flows is open in the space of C^1 flows, the induced flow Φ_ρ is Anosov if $\rho \in \mathcal{A}_*(M_\Gamma, P)$ is sufficiently C^1-close to ρ_0. In this case, E_ρ^- is the strong stable subbundle of Φ_ρ and the Anosov flow Φ_ρ is s-conformal with respect to g_ρ.

3.2 Reduction to the conjugacy of induced flows

We reduce Theorem 1.1 to the smooth conjugacy problem of the induced flows.

Theorem 3.3. *Let ρ be a locally free action in $\mathcal{A}_*(M_\Gamma, P)$. Suppose that a C^∞ diffeomorphism h of N_Γ satisfies $\Phi_\rho^t \circ h = h \circ \Phi_{\rho_0}^t$ for any $t \in \mathbb{R}$. Then, ρ is C^∞-conjugate to the standard P-action ρ_0.*

Let $\operatorname{Fr} E_\rho^-$ be the frame bundle of E_ρ^-. It admits a natural right action of $GL(n-1, \mathbb{R})$. The flow Φ_ρ induce a flow $\operatorname{Fr} \Phi_\rho$ on $\operatorname{Fr} E_\rho^-$. Let OE_ρ^- be the orthonormal frame bundle of (E_ρ^-, g_ρ). We define a map $\psi_\rho : M_\Gamma \to OE_\rho^-$ by

$$\psi_\rho(x) = (D\pi(Y_1^\rho(x)), \ldots, D\pi(Y_{n-1}(x))).$$

By Lemma 3.2, $OE_\rho^-(y) = \{\psi_\rho(x) \mid x \in \pi^{-1}(y)\}$ for any $y \in N_\Gamma$ and ψ_ρ is a diffeomorphism from M_Γ to OE_ρ^-. By Equation (1), we have $\operatorname{Fr} \Phi_\rho^t(\psi(x)) = e^{-t}\psi_\rho(\rho^{\exp(tX)}(x))$. Hence, we can define a flow $O\Phi_\rho$ on OE_ρ^- by

$$O\Phi_\rho^t(\psi_\rho(x)) = e^t \cdot \operatorname{Fr} \Phi_\rho^t(\psi_\rho(x)) = \psi_\rho(\rho^{\exp(tX)}(x)).$$

In particular, the map ψ_ρ is a C^∞ conjugacy between $\rho^{\exp(tX)}$ and $O\Phi_\rho^t$. By Moore's ergodicity theorem, the flow $(\rho^{\exp(tX)})_{t \in \mathbb{R}}$ is topologically transitive. Hence, so the flow $O\Phi_\rho$ is.

Fix $\rho \in \mathcal{A}_*(M_\Gamma, P)$ and suppose that there exists a C^∞ diffeomorphism h of N_Γ such that

$$\Phi_\rho^t \circ h = h \circ \Phi_{\rho_0}^t \tag{5}$$

for any $t \in \mathbb{R}$.

Lemma 3.4. *$Dh(E_{\rho_0}^-) = E_\rho^-$ and there exists a constant $c_h > 0$ such that $\|Dh(v)\|_{g_\rho} = c_h \cdot \|v\|_{g_\Gamma}$ for any $v \in E_{\rho_0}^-$.*

Proof. Recall that the flow Φ_{ρ_0} is Anosov and $E_{\rho_0}^-$ is its strong stable subbundle. Since h is a C^∞ conjugacy between Φ_{ρ_0} and Φ_ρ, the flow Φ_ρ is also Anosov and its strong stable subbundle is $Dh(E_{\rho_0}^-)$. By Equation (3), the subbundle E_ρ^- is contained in the strong stable subbundle $Dh(E_{\rho_0}^-)$. Since their dimensions are equal, we have $Dh(E_{\rho_0}^-) = E_\rho^-$.

Let $SE_{\rho_0}^-$ be the unit sphere bundle $\{v \in E_{\rho_0}^- \mid \|v\|_{g_{\rho_0}} = 1\}$ of $E_{\rho_0}^-$ and $\pi_O : OE_{\rho_0}^- \to SE_{\rho_0}^-$ be the projection defined by $(v_1, \ldots, v_{n-1}) \mapsto v_1$. By Equation (3) for ρ_0, we can define a flow $S\Phi_{\rho_0}$ on $SE_{\rho_0}^-$ by $S\Phi_{\rho_0}^t = e^t D\Phi_{\rho_0}^t$. Then, $\pi_O \circ O\Phi_{\rho_0}^t = S\Phi_{\rho_0}^t \circ \pi_O$. Since $O\Phi_{\rho_0}$ is topologically transitive, $S\Phi_{\rho_0}$ also is. Take $v_0 \in SE_{\rho_0}^-$ such that the orbit $\{S\Phi_{\rho_0}^t(v_0) \mid t \in \mathbb{R}\}$ is dense in $SE_{\rho_0}^-$. Put $c_h = \|Dh(v_0)\|_{g_\rho}$. By Equation (3),

$$\begin{aligned}
\|Dh \circ S\Phi_{\rho_0}^t(v_0)\|_{g_\rho} &= e^t \|Dh \circ D\Phi_{\rho_0}^t(v_0)\|_{g_\rho} \\
&= e^t \|D\Phi_\rho^t \circ Dh(v_0)\|_{g_\rho} \\
&= \|Dh(v_0)\|_{g_\rho} \\
&= c_h
\end{aligned}$$

for any $t \in \mathbb{R}$. It implies that $\|Dh(v)\|_{g_\rho} = c_h$ for any $v \in SE_{\rho_0}^-$. ∎

Proposition 3.5. *There exists a C^∞ diffeomorphism H of M_Γ such that $H \circ \rho_0^g = \rho^g \circ H$ for any $g \in \{\exp(tX)m \mid t \in \mathbb{R}, m \in SO(n-1)\}$.*

Proof. Let $\mathrm{Fr}\, h$ be the lift of h to $\mathrm{Fr}\, E^-_{\rho_0}$. By the above lemma, we can define a diffeomorphism $Oh : OE^-_{\rho_0} \to OE^-_\rho$ by $Oh = c_h^{-1}\,\mathrm{Fr}\, h$. Then, $O\Phi^t_\rho \circ Oh = Oh \circ O\Phi^t_{\rho_0}$ for any $t \in \mathbb{R}$. Since $\mathrm{Fr}\, h$ commutes with the action of $SO(n-1)$, we have $Oh(z \cdot m) = Oh(z) \cdot m$ for any $z \in OE^-_{\rho_0}$ and $m \in SO(n-1)$. Put $H = \psi_\rho^{-1} \circ Oh \circ \psi_{\rho_0}$. Then, we have

$$\begin{aligned} H \circ \rho_0^{\exp(tX)} &= \rho^{\exp(tX)} \circ H, \\ H(x \cdot m) &= H(x) \cdot m \end{aligned}$$

for any $t \in \mathbb{R}$ and any $m \in SO(n-1)$. $\square$

The proof of Theorem 3.3 will finish once we show the following

Proposition 3.6. *There exists an automorphism θ of P such that $\rho(H(x), \theta(g)) = \rho_0(x, g)$ for any $x \in M_\Gamma$ and $g \in P$.*

Proof. Since $\rho^{\exp(tX)} \circ H = H \circ \rho_0^{\exp(tX)}$ for any $t \in \mathbb{R}$, we have

$$\begin{aligned} D\rho^{\exp(tX)}(DH(Y_j^{\rho_0}(x))) &= DH(D\rho_0^{\exp(tX)}(Y_j^{\rho_0}(x))) \\ &= e^{-t} DH(Y_j^{\rho_0}(\rho_0^{tX}(x))). \end{aligned} \tag{6}$$

We also have

$$\begin{aligned} DH(\langle Y_1^{\rho_0}(x), \cdots, Y_{n-1}^{\rho_0}(x)\rangle) &= DH(\{v \in T_x M_\Gamma \mid \lim_{t\to+\infty} \|D\rho_0^{\exp(tX)}(v)\| = 0\}) \\ &= \{v' \in T_{H(x)} M_\Gamma \mid \lim_{t\to+\infty} \|D\rho^{\exp(tX)}(v')\| = 0\} \\ &\supset \langle Y_1^\rho(H(x)), \cdots, Y_{n-1}^\rho(H(x))\rangle \end{aligned}$$

for any $x \in M_\Gamma$. In particular,

$$DH(\langle Y_1^{\rho_0}(x), \cdots, Y_{n-1}^{\rho_0}(x)\rangle) = \langle Y_1^\rho(H(x)), \cdots, Y_{n-1}^\rho(H(x))\rangle.$$

Since the flow $(\rho_0^{\exp(tX)})_{t\in\mathbb{R}}$ is topologically transitive, there exists $x_0 \in M_\Gamma$ such that $\{\rho_0^{tX}(x_0) \mid t \in \mathbb{R}\}$ is a dense subset of M_Γ. Let $b = (b_{ij})_{i,j=1,\cdots,n-1}$ be the square matrix given by $Y_j^\rho(H(x_0)) = \sum_{i=1}^{n-1} b_{ij} DH(Y_i^{\rho_0}(x_0))$. Remark that it is an invertible matrix. For any $t \in \mathbb{R}$, we have

$$\begin{aligned} Y_j^\rho(H(\rho_0^{\exp(tX)}(x_0))) &= Y_j^\rho(\rho^{\exp(tX)}(H(x_0))) \\ &= e^t D\rho^{\exp(tX)}(Y_j^\rho(H(x_0))) \\ &= \sum_{i=1}^{n-1} b_{ij} DH(Y_i^{\rho_0}(\rho_0^{\exp(tX)}(x_0))). \end{aligned}$$

Since the orbit of x_0 is dense, $Y_j^{\rho}(H(x)) = \sum_{i=1}^{n-1} b_{ij} DH(Y_i^{\rho_0}(x))$ for any $x \in M_\Gamma$. In particular,

$$DH(Y_1^{\rho_0}, \dots, Y_{n-1}^{\rho_0}) = (Y_1^{\rho}, \dots, Y_{n-1}^{\rho}) \cdot b^{-1} \tag{7}$$

Recall that

$$D\rho_0^m(Y_1^{\rho_0}, \dots, Y_{n-1}^{\rho_0}) = (Y_1^{\rho_0}, \dots, Y_{n-1}^{\rho_0}) \cdot m,$$
$$D\rho^m(Y_1^{\rho}, \dots, Y_{n-1}^{\rho}) = (Y_1^{\rho}, \dots, Y_{n-1}^{\rho}) \cdot m$$

for any $m \in SO(n-1)$. Since $\rho^m \circ H = H \circ \rho_0^m$ for any m, Equation (7) implies

$$(Y_1^{\rho_0}, \dots, Y_{n-1}^{\rho_0}) \cdot mb^{-1} = (Y_1^{\rho_0}, \dots, Y_{n-1}^{\rho_0}) \cdot b^{-1}m.$$

Hence, b commutes with any $m \in SO(n-1)$. It is easy to check that

- if $n \geq 4$, then there exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that $b = \alpha I_{n-1}$, where I_{n-1} is the unit matrix of size $(n-1)$,
- if $n = 3$, then there exists $\alpha \in \mathbb{R} \setminus \{0\}$ and $m_0 \in SO(2)$ such that such that $b = \alpha m_0$.

In each case, $\alpha^{-1}b$ is contained in the center of $SO(n-1)$.

We define a map $\theta : P \to SO_+(n,1)$ by $\theta(g) = bgb^{-1}$. Since $\alpha^{-1}b$ is contained in the center of $SO(n-1)$, we have $\theta(P) = P$. In particular, θ is an automorphism of P such that

$$\theta_*(Y_1, \dots, Y_{n-1}) = (Y_1, \dots, Y_{n-1}) \cdot b^{-1},$$

where θ_* is the induced automorphism of the Lie algebra of P. By Equation (7), $\rho(H(x), \theta(\exp(Y_i)) = H(\rho_0(x, \exp(Y_i)))$ for any $x \in M_\Gamma$ and $i = 1, \dots, n-1$. On the other hand, $\theta(g') = g'$ and $\rho^{g'} \circ H = H \circ \rho_0^{g'}$ for any $g' \in \{\exp(tX)m \mid t \in \mathbb{R}, m \in SO(n-1)\}$. Therefore, $\rho(H(x), \theta(g) = H(\rho_0(x, g))$ for any $x \in M_\Gamma$ and $g \in P$. $\square$

3.3 Smooth conjugacy between induced flows

In this subsection, we show the following theorem. With Proposition 3.1 and Theorem 3.3, it completes the proof of the main theorem.

Theorem 3.7. *If $\rho \in \mathcal{A}_*(M_\Gamma, P)$ is sufficiently C^∞-close to ρ_0, then there exists a C^∞ diffeomorphism h of N_Γ such that $\Phi_\rho^t \circ h = h \circ \Phi_{\rho_0}^t$ for any $t \in \mathbb{R}$.*

Choose $\rho \in \mathcal{A}_*(M_\Gamma, P)$ such that Φ_ρ is an s-conformal Anosov flow with respect to g_ρ and E_ρ^- is transverse to $T\Phi_{\rho_0} \oplus E_{\rho_0}^+$. By $\mathcal{F}_{\rho_0}^{ss}$, $\mathcal{F}_{\rho_0}^{s}$, $\mathcal{F}_{\rho_0}^{uu}$, $\mathcal{F}_{\rho_0}^{u}$, we denote the strong stable, strong unstable, weak stable, weak unstable foliations of Φ_{ρ_0}, respectively. Similarly, by $\mathcal{F}_{\rho}^{ss}$, $\mathcal{F}_{\rho}^{s}$, we denote the strong stable and weak stable foliations of Φ_ρ, respectively. Remark that all of them are C^∞ foliations, but the strong unstable and weak unstable foliations of Φ_ρ may not be C^∞.

By X_{ρ_0} and X_ρ, we denote the vector fields generating the flows Φ_{ρ_0} and Φ_ρ. Let $\sigma_1 : TN_\Gamma \to E^+_{\rho_0}$ and $\sigma_2 : TN_\Gamma \to E^-_\rho$ be the projection with respect to the splittings $E^+_{\rho_0} \oplus (T\Phi_\rho \oplus E^-_\rho)$ and $E^-_\rho \oplus (T\Phi_{\rho_0} \oplus E^+_{\rho_0})$ of TN_Γ, respectively. Put $X_1 = X_{\rho_0} - \sigma_1(X_{\rho_0})$ and $X_2 = X_\rho - \sigma_2(X_\rho)$. They generates flows Ψ_1 and Ψ_2 on N_Γ. If ρ is sufficiently C^∞-close to ρ_0, then Ψ_1 and Ψ_2 are C^∞-close to Φ_{ρ_0}. Hence, we may assume that they are Anosov flows. Since $[X_1, E^+_{\rho_0}] \subset E^+_{\rho_0}$ and $X_1 \in T\Phi_\rho \oplus E^-_\rho$, we have

$$\Psi^t_1(x) \in \mathcal{F}^{uu}_{\rho_0}(\Phi^t_{\rho_0}(x)) \cap \mathcal{F}^s_\rho(x). \tag{8}$$

for any $x \in M_\Gamma$ and $t \in \mathbb{R}$. If ρ is sufficiently close to ρ_0, then $D\Psi_1$ expands $E^+_{\rho_0}$ uniformly. So, we may assume that $E^+_{\rho_0}$ is the strong unstable subbundle of Ψ_1. Similarly, we may assume

$$\Psi^t_2(x) \in \mathcal{F}^{ss}_\rho(\Phi^t_\rho(x)) \cap \mathcal{F}^u_{\rho_0}(x) \tag{9}$$

for any $x \in N_\Gamma$ and $t \in \mathbb{R}$, and E^-_ρ is the strong stable subbundle of Ψ_2. Since both $\Psi^t_1(x)$ and $\Psi^t_2(x)$ are contained in $\mathcal{F}^u_{\rho_0}(x) \cap \mathcal{F}^s_\rho(x)$, the orbits of Ψ_1 and Ψ_2 coincide. Hence, there exists a C^∞ cocycle over Ψ_2 such that

$$\Psi^t_2(x) = \Psi^{\alpha(x,t)}_1(x) \tag{10}$$

for any $x \in N_\Gamma$ and $t \in \mathbb{R}$. Since each leaf of $\mathcal{F}^u_{\rho_0}$ is Φ_{ρ_0}- and Φ_ρ-invariant and it is transverse to both $E^-_{\rho_0}$ and E^-_ρ, we have

$$\det D\Psi^{\alpha(x,T)}_1|_{E^-_{\rho_0}(x)} = \det D\Psi^T_2|_{E^-_\rho(x)} \tag{11}$$

for any $(x,T) \in N_\Gamma \times \mathbb{R}$ satisfying $\Psi_2(x)^T = x$.

By Proposition 2.1, there exist a homeomorphism h_1 of N_Γ such that $\Psi^t_1 \circ h_1 = h_1 \circ \Phi^t_{\rho_0}$ for any $t \in \mathbb{R}$ and $h_1(\mathcal{F}^{uu}_{\rho_0}(x)) = \mathcal{F}^{uu}_{\rho_0}(x)$ for any $x \in N_\Gamma$. Since h_1 preserves each leaf of $\mathcal{F}^u_{\rho_0}$, we have

$$\det D\Psi^{T'}_1|_{E^-_\rho(x)} = \det D\Phi^{T'}_{\rho_0}|_{E^-_{\rho_0}(h_1^{-1}(x))} = e^{-(n-1)T'} \tag{12}$$

for any $(x,T') \in N_\Gamma \times \mathbb{R}$ satisfying $\Psi^{T'}_1(x) = x$.

By Proposition 2.1 again, there exist a homeomorphism h_2 of N_Γ such that $\Psi^t_2 \circ h_2 = h_2 \circ \Phi^t_\rho$ for any $t \in \mathbb{R}$ and $h(\mathcal{F}^{ss}_\rho(x)) = \mathcal{F}^{ss}_\rho(x)$ for any $x \in N_\Gamma$. Since $\mathcal{F}^u_{\rho_0}$ is a transversely conformal foliation, Ψ_2 is s-conformal. By Theorem 2.3, the restriction of h_2 to each leaf of $\mathcal{F}^{ss}_\rho$ is smooth. Hence,

$$\det D\Psi^T_2|_{E^-_\rho(x)} = \det D\Phi^T_\rho|_{E^-_\rho(h_2^{-1}(x))} = e^{-(n-1)T} \tag{13}$$

for any $(x,T) \in N_\Gamma \times \mathbb{R}$ satisfying $\Psi^T_2(x) = x$.

By Equations (11), (12), and (13), we have $\alpha(x,T) - T = 0$ for any $(x,T) \in N_\Gamma \times \mathbb{R}$ satisfying $\Psi^T_2(x) = x$. By The C^∞ Livschitz Theorem, there exists a C^∞ function β on N_Γ such that

$$\alpha(x,t) - t = \beta(\Psi^t_2(x)) - \beta(x)$$

for any $x \in N_\Gamma$ and $t \in \mathbb{R}$. We define a map $h_3 : N_\Gamma \to N_\Gamma$ by $h_3(x) = \Psi_2^{-\beta(x)}$. Remark that if ρ is sufficiently C^∞-close to ρ_0, then α is C^∞-close to zero, and hence, we can choose β so that it is C^∞-close to 0. So, we may assume that h_3 is a C^∞ diffeomorphism sufficiently C^∞-close to the identity map. Since Φ_{ρ_0} is a contact Anosov flow and the set of contact structures is open in the space of C^1 hyperplane fields, we also may assume that $Dh_3(E_\rho^-) \oplus E_{\rho_0}^+$ is a contact structure.

By Equation (10), we have $\Psi_1^t \circ h_3 = h_3 \circ \Psi_2^t$. In particular, $Dh_3(E_\rho^-)$ is the strong stable subbundle of Ψ_1. Since $E_{\rho_0}^+$ is the strong unstable subbundle of Ψ_1, the flow Ψ_1 is a contact Anosov flow. Since Ψ_1 is s-conformal, it is also u-conformal by Proposition 2.4. By Theorem 2.3, h_1 is a C^∞ diffeomorphism.

Since $\mathcal{F}_{\rho_0}^s$ is a transversely conformal foliation, $\mathcal{F}_\rho^s = h_1(\mathcal{F}_{\rho_0}^s)$ also is. It implies that Φ_ρ and Ψ_2 are u-conformal. Since Φ_ρ and Ψ_2 are s-conformal, Theorem 2.3 implies that h_2 is a C^∞ diffeomorphism. Now, we put $h = h_2^{-1} \circ h_3^{-1} \circ h_1$. Then, h is a C^∞ diffeomorphism and $\Phi_\rho^t \circ h = h \circ \Phi_{\rho_0}^t$ for any $t \in \mathbb{R}$. The proof of Theorem 3.7 is completed.

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